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GENERALIZED BRAIDED QUANTUM GROUPS

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ABSTRACT

A generalization of Hopf algebras (quantum groups), and braided-Hopf algebras (braided quantum groups) in which the multiplicativity axiom for the counit is dropped, is presented. The generalization overcomes an inherent geometrical inhomogeneity of standard quantum groups and braided quantum groups, in the sense of allowing completely 'pointless' objects. All braid-type equations appear as a consequence of deeper axioms. Braided counterparts of basic algebraic relations between fundamental entities of the standard theory are found.

1. Introduction

The aim of this study is to present basic elements of a braided theory which generalizes standard quantum groups and braided quantum groups in a non-trivial and effective way.

The theory allows the possibility of completely 'pointless' objects and includes, besides standard braided quantum groups, various geometrically interesting structures which are not braided-Hopf algebras, but which are more or less similar to them.

Let us start with a simple geometrical consideration. According to the classical Gelfand–Naimark theorem, there exists a natural correspondence between compact topological spaces X and commutative unital C^*-algebras $\mathcal{A}$. For a given X, the algebra $\mathcal{A}$ consists of complex-valued continuous functions on X, endowed with the standard algebraic operations and the maximum norm. Conversely, if $\mathcal{A}$ is given then points of X are recovered as characters (non-trivial multiplicative

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hermitian linear functionals) on $\mathcal{A}$. In terms of this identification, topology of X is induced by the *-weak topology of the dual space.

Furthermore, in differential geometry it is possible to re-express all properties of a smooth manifold X in terms of the *-algebra of smooth complex-valued functions on X. A similar situation holds in algebraic geometry, where $\mathcal{A}$ consists of polynomial functions on the algebraic variety X.

The starting idea of non-commutative differential geometry [3] consists in replacing function algebras by appropriate non-commutative algebras $\mathcal{A}$, but still interpreting the elements of $\mathcal{A}$ as 'functions' on the qualitatively new 'quantum spaces'. In non-commutative geometry, the 'existence' of such 'quantum spaces' always appears through $\mathcal{A}$. In other words, we work directly with the algebra $\mathcal{A}$, and all geometrical concepts and structures are expressed exclusively in terms of the algebra $\mathcal{A}$. This means that formally we define quantum spaces as ordered pairs $X = (\mathcal{A}, S)$, where S is the appropriate additional algebraic structure on $\mathcal{A}$, corresponding in the classical (commutative) case to the appropriate geometrical structure on the classical underlying space X.

However, if $\mathcal{A}$ is non-commutative, then the corresponding quantum space X cannot be re-interpreted in classical terms as a structuralized collection of points. On the other hand, it is important to notice that the concept of a classical point is easily incorporable in the non-commutative context. In analogy with the classical geometry, it is natural to define points of X as characters of the algebra $\mathcal{A}$, assuming that $\mathcal{A}$ is equipped with a *-structure (if $\mathcal{A}$ is not equipped with a *-structure we can simply consider all multiplicative functionals, in analogy with complex algebraic geometry). Generally, the space X may be 'completely quantum'—without points at all.

A particularly important class of quantum spaces is given by quantum groups. Geometrically, quantum groups are quantum spaces endowed with a group structure. Let us consider a quantum group $G = (\mathcal{A}, S)$. By definition, this means that $S = \{\phi, \epsilon, \kappa\}$ is a Hopf algebra [1] structure on the algebra $\mathcal{A}$, specified by the coproduct $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, the counit $\epsilon\colon \mathcal{A} \to \mathbb{C}$ and the antipode $\kappa\colon \mathcal{A} \to \mathcal{A}$ map (we follow the notation of [7]). The three maps should be mutually related in the following way.

Firstly, the maps ϕ and ϵ determine a counital coalgebra structure on $\mathcal{A}$; in other words

$$(\mathrm{id} \otimes \phi)\phi = (\phi \otimes \mathrm{id})\phi, \qquad (\mathrm{id} \otimes \epsilon)\phi = (\epsilon \otimes \mathrm{id})\phi = \mathrm{id}.$$

Secondly, we have the antipode axiom

$$m(\kappa \otimes \mathrm{id})\phi = m(\mathrm{id} \otimes \kappa)\phi = 1\epsilon,$$

where $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication in $\mathcal{A}$ and $1 \in \mathcal{A}$ is the unit element.

Finally, the map ϕ should be multiplicative, in the sense that

$$\phi(ab) = \phi(a)\phi(b),$$

for each $a, b \in \mathcal{A}$. In the above relation, $\mathcal{A} \otimes \mathcal{A}$ is understood as an algebra, in a natural manner.

As a consequence of the mentioned properties, it turns out that the antipode κ is an anti(co)multiplicative map. The multiplicativity of the counit is another important consequence. Further, if $\mathcal{A}$ is equipped with a *-structure and if the coproduct is such that $\phi * = (* \otimes *)\phi$, then the composition $*\kappa$ is involutive and the counit is hermitian.

In particular, the space G always possesses at least one point, corresponding to the counit map (the neutral element). The quantum group structure on G induces, in a natural manner, a group structure on the set G_{cl} of all classical points of G, such that G_{cl} is geometrically interpretable as a 'subgroup' of G. Explicitly, the product and the inverse are given by

$$gh = (g \otimes h)\phi, \qquad g^{-1} = g\kappa.$$

In this sense, quantum groups are 'inhomogeneous' objects. This inhomogeneity explicitly shows up in certain geometrical constructions [4]. On the other hand, it is natural to expect that in noncommutative geometry quantum spaces with a group structure play a similar role as Lie groups in classical differential geometry. As such, they should be particularly regular geometrical objects and not forced to have this 'inhomogeneity' in which part is classical and part is purely quantum.

Such thinking naturally leads to the idea of generalizing the notion of a group structure on a noncommutative space, in order to include objects of a more elaborate geometrical nature.

There has already been introduced in [6] one generalization of quantum groups, in the framework of braided categories. In this generalization, the standard transposition (figuring in the product in $\mathcal{A} \otimes \mathcal{A}$) is replaced by the appropriate braid operator $\sigma\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$, so that all group entities are undersandable as morphisms in the braided category generated by $\mathcal{A}$ and σ. Such a generalization has proven useful for various applications in non-commutative geometry, however it does not address the above inhomogeneity question because the counit map is always multiplicative.

In this paper, we further generalize this concept of a braided-Hopf algebra (by not demanding that the counit is multiplicative) replacing the standard axiom of the σ-functoriality of the coproduct ϕ by a more general octagonal diagram. This introduces the possibility of the existence of 'completely pointless' structures (in particular, in this case the counit is not multiplicative). Moreover, we shall not demand directly that σ obeys the braid equation, though this will be derived as a consequence of the initial axioms.

The paper is organized as follows. The next section is devoted to the definition of braided quantum groups. In Section 3 the most important interrelations between all relevant maps will be investigated. In particular, we shall see that besides the flip-over operator σ, another braid operator $\tau\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ naturally enters the game. This operator is expressible via ϵ, ϕ and σ. Two braid operators σ and τ will play a fundamental role in the whole analysis. In particular, it will be shown that σ and τ are mutually compatible in a 'braided sense'.

The standard theory of braided quantum groups [6] is recovered when $\sigma=\tau$. Interestingly, this is further equivalent to the multiplicativity of the counit map.

A large class of examples of 'completely pointless' braided quantum groups is given by braided Clifford algebras [5] associated to involutive braidings. This includes classical Clifford and Weyl algebras. Another class of interesting examples is given by quantum tori [3]. Endowed with appropriate group structures [2], quantum tori can be viewed as braided quantum groups, in the sense of the formalism presented in this paper.

Finally, the Appendix is devoted to the main properties of systems of braid operators, mutually compatible in a 'braided sense'. The motivation for this comes from the already mentioned braided compatibility between σ and τ. In particular, it will be shown that σ and τ can be naturally included in a (generally infinite) 'braid system' expressing concisely all twisting properties.

2. Definition of braided quantum groups

Let $\mathcal{A}$ be a complex associative algebra, with the product $m\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ and the unit element $1\in\mathcal{A}$. Let us assume that $\mathcal{A}$ is endowed with a coassociative coalgebra structure, specified by the coproduct $\phi\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ and the counit $\epsilon\colon \mathcal{A}\to\mathbb{C}$. Finally, let us assume that bijective linear maps $\kappa\colon \mathcal{A}\to\mathcal{A}$ and $\sigma\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ are given such that the following equalities hold:

(1) $$\sigma(m\otimes\mathrm{id}) = (\mathrm{id}\otimes m)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma),$$

(2) $$\sigma(\mathrm{id}\otimes m) = (m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id}),$$

(3) $$\phi m = (m\otimes m)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\phi\otimes\phi),$$

(4) $$(\sigma\otimes\mathrm{id}^2)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi) = (\mathrm{id}^2\otimes\sigma)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1})(\phi\otimes\mathrm{id}),$$

together with the antipode axiom

(5) $$1\epsilon = m(\mathrm{id}\otimes\kappa)\phi = m(\kappa\otimes\mathrm{id})\phi.$$

Definition 1: Every pair $G=\bigl(\mathcal{A},\{\phi,\epsilon,\kappa,\sigma\}\bigr)$ satisfying the above requirements is called a **braided quantum group**.

The map σ is interpretable as the 'twisting operator'. In the standard theory, σ reduces to the ordinary transposition. Identities (1)–(4) express mutual compatibility between maps ϕ, m and σ. It is important to mention that the asymmetry between (1)–(2) and (4) implies that the theory is not 'selfdual'. However, if we replace (4) with 'dual' counterparts of (1)–(2) then the theory reduces to braided quantum groups of [6] (and, in particular, becomes selfdual).

The space $\mathcal{A}\otimes\mathcal{A}$ is an $\mathcal{A}$-bimodule, in a natural manner. With the help of σ, a natural product can be defined on $\mathcal{A}\otimes\mathcal{A}$ by requiring

(6) $$(a\otimes b)(c\otimes d) = a\sigma(b\otimes c)d.$$

Identities (1)–(2) ensure that this defines an associative algebra structure on $\mathcal{A}\otimes\mathcal{A}$, such that $1\otimes 1$ is the unit element. In particular,

(7) $$\sigma\bigl(1\otimes(\,)\bigr) = (\,)\otimes 1, \qquad \sigma\bigl((\,)\otimes 1\bigr) = 1\otimes(\,).$$

In the following, it will be assumed that $\mathcal{A}\otimes\mathcal{A}$ is endowed with this algebra structure. Equality (3) then says that ϕ is multiplicative.

Identity (4) expresses the coassociativity of the map $(\mathrm{id}\otimes\sigma^{-1}\otimes\mathrm{id})(\phi\otimes\phi)$. The 'inverse' identity

$$\begin{aligned}&(\mathrm{id}^2\otimes\sigma)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\\ =&(\sigma\otimes\mathrm{id}^2)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\phi\otimes\mathrm{id})\end{aligned}\tag{8}$$

holds, too. It expresses the coassociativity of $(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\phi\otimes\phi)$.

3. Elementary algebraic properties

Let $G=\big(\mathcal{A},\{\phi,\epsilon,\kappa,\sigma\}\big)$ be a braided quantum group. As in the standard theory, the antipode is uniquely determined by (5). The flip-over operator σ is expressible through ϕ, m and κ in the following way:

$$\sigma=(m\otimes m)(\kappa\otimes\phi m\otimes\kappa)(\phi\otimes\phi),\tag{9}$$

as directly follows from (3) and (5).

It is easy to see that

$$\phi(1)=1\otimes 1.\tag{10}$$

Indeed, $\phi(1)$ is the unity in the subalgebra $\phi(\mathcal{A})\subseteq\mathcal{A}\otimes\mathcal{A}$, as follows from (3). On the other hand, $\mathcal{A}\otimes\mathcal{A}$ is generated by $\phi(\mathcal{A})$, as a left (right) $\mathcal{A}$-module. Hence, $\phi(1)$ is the unity of $\mathcal{A}\otimes\mathcal{A}$. From (10) we obtain

$$\epsilon(1)=1,\tag{11}$$

$$\kappa(1)=1.\tag{12}$$

In further computations the result of an $(n-1)$-fold comultiplication of an element $a\in\mathcal{A}$ will be symbolically denoted by $a^{(1)}\otimes\ldots\otimes a^{(n)}$. Clearly, this element of $\mathcal{A}$ is independent of ways in which the corresponding comultiplications are performed.

LEMMA 1: *The following identities hold:*

$$(\epsilon\otimes\mathrm{id})=(\mathrm{id}\otimes\epsilon m)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\phi),\tag{13}$$

$$(\mathrm{id}\otimes\epsilon)=(\epsilon m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\phi\otimes\mathrm{id}).\tag{14}$$

Proof: According to (3),

$$ab^{(1)}\otimes b^{(2)}=(\epsilon\otimes\mathrm{id})(m\otimes m)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})\big(a^{(1)}\otimes a^{(2)}\otimes b^{(1)}\otimes b^{(2)}\big)\otimes b^{(3)},$$

for each $a, b \in \mathcal{A}$. Acting by $m(\mathrm{id} \otimes \kappa)$ on this equality, and using (5), we obtain

$$a\epsilon(b) = (\epsilon \otimes \mathrm{id})\big(a^{(1)}\sigma(a^{(2)} \otimes b)\big).$$

Similarly, acting by $m(\kappa \otimes \mathrm{id})$ on the identity

$$a^{(1)} \otimes a^{(2)}b = a^{(1)} \otimes (\mathrm{id} \otimes \epsilon)(m \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(a^{(2)} \otimes a^{(3)} \otimes b^{(1)} \otimes b^{(2)})$$

we obtain

$$\epsilon(a)b = (\mathrm{id} \otimes \epsilon)\big(\sigma(a \otimes b^{(1)})b^{(2)}\big). \qquad \blacksquare$$

A 'secondary' flip-over operator τ will now be introduced in the game. From (8) we obtain

$$(\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes \sigma^{-1})(\phi \otimes \mathrm{id}) = (\epsilon \otimes \mathrm{id}^2)(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi). \tag{15}$$

Let $\tau \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by

$$\tau = (\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes \sigma^{-1})(\phi \otimes \mathrm{id})\sigma = (\epsilon \otimes \mathrm{id}^2)(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma. \tag{16}$$

LEMMA 2: *The map τ is bijective and*

$$\tau^{-1}\sigma = (\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes \sigma)(\phi \otimes \mathrm{id}) = (\epsilon \otimes \mathrm{id}^2)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi). \tag{17}$$

Proof: The second equality in (17) follows from (4). Let $\tau'\sigma$ be the map given by the second term in (17). A direct computation gives

$$\begin{aligned}
\tau\tau'\sigma &= (\epsilon \otimes \mathrm{id}^2 \otimes \epsilon)(\sigma^{-1} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\phi \otimes \mathrm{id})\\
&= (\epsilon \otimes \mathrm{id}^2 \otimes \epsilon)(\sigma^{-1} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma \otimes \epsilon \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)\\
&\qquad (\phi \otimes \mathrm{id})\\
&= (\epsilon \otimes \mathrm{id}^2 \otimes \epsilon \otimes \epsilon)(\sigma^{-1} \otimes \mathrm{id} \otimes \sigma)(\mathrm{id} \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma \otimes \mathrm{id})\\
&\qquad (\mathrm{id} \otimes \phi)\\
&= (\epsilon \otimes \mathrm{id}^2 \otimes \epsilon \otimes \epsilon)(\mathrm{id}^3 \otimes \sigma)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma^{-1})(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)\\
&\qquad (\phi \otimes \mathrm{id})\\
&= (\mathrm{id}^2 \otimes \epsilon \otimes \epsilon)(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\phi \otimes \mathrm{id})\sigma\\
&= (\mathrm{id}^2 \otimes \epsilon \otimes \epsilon)(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma\\
&= \sigma.
\end{aligned}$$

Similarly, interchanging σ and σ^{-1} in the above computations we conclude that τ' is a left inverse for τ. Hence, τ is bijective and $\tau^{-1}=\tau'$. ∎

Let us write down some important algebraic relations including the map τ. Initially, let us observe that

$$(\epsilon\otimes\mathrm{id})\tau=\mathrm{id}\otimes\epsilon,\qquad (\mathrm{id}\otimes\epsilon)\tau=\epsilon\otimes\mathrm{id}, \tag{18}$$

$$\tau\bigl(1\otimes(\,)\bigr)=(\,)\otimes 1,\qquad \tau\bigl((\,)\otimes 1\bigr)=1\otimes(\,). \tag{19}$$

This is a direct consequence of the definition of τ, and property (7). Further, coassociativity of ϕ and relations (16)–(17) imply

$$(\phi\otimes\mathrm{id})\tau^{-1}\sigma=(\mathrm{id}\otimes\tau^{-1}\sigma)(\phi\otimes\mathrm{id}), \tag{20}$$

$$(\mathrm{id}\otimes\phi)\tau^{-1}\sigma=(\tau^{-1}\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\phi), \tag{21}$$

$$(\phi\otimes\mathrm{id})\tau\sigma^{-1}=(\mathrm{id}\otimes\tau\sigma^{-1})(\phi\otimes\mathrm{id}), \tag{22}$$

$$(\mathrm{id}\otimes\phi)\tau\sigma^{-1}=(\tau\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi). \tag{23}$$

In other words, maps $\sigma\tau^{-1}$ and $\sigma^{-1}\tau$ are automorphisms of the $\mathcal{A}$-bicomodule $\mathcal{A}\otimes\mathcal{A}$ (with the left and the right $\mathcal{A}$-comodule structures given by $\phi\otimes\mathrm{id}$ and $\mathrm{id}\otimes\phi$ respectively). Moreover, the following commutation relations hold:

$$(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma\tau^{-1})=(\mathrm{id}\otimes\sigma\tau^{-1})(\sigma\tau^{-1}\otimes\mathrm{id}), \tag{24}$$

$$(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1}\tau)=(\mathrm{id}\otimes\sigma^{-1}\tau)(\sigma\tau^{-1}\otimes\mathrm{id}), \tag{25}$$

$$(\sigma^{-1}\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma\tau^{-1})=(\mathrm{id}\otimes\sigma\tau^{-1})(\sigma^{-1}\tau\otimes\mathrm{id}), \tag{26}$$

$$(\sigma^{-1}\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1}\tau)=(\mathrm{id}\otimes\sigma^{-1}\tau)(\sigma^{-1}\tau\otimes\mathrm{id}). \tag{27}$$

The above equalities follow from (20)–(23) and (16)–(17). As a direct consequence of Lemma 1 and (17) we find

$$\epsilon m=(\epsilon\otimes\epsilon)\sigma^{-1}\tau. \tag{28}$$

This generalizes the standard multiplicativity law for the counit.

Identities (4) and (8) can be rewritten in a simpler 'pentagonal form', including the operator τ and explicitly expressing twisting properties of the coproduct map.

PROPOSITION 3: *The following identities hold:*

$$(\phi \otimes \mathrm{id})\sigma = (\mathrm{id} \otimes \tau)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi), \tag{29}$$

$$(\mathrm{id} \otimes \phi)\sigma = (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\phi \otimes \mathrm{id}), \tag{30}$$

$$(\phi \otimes \mathrm{id})\sigma = (\mathrm{id} \otimes \sigma)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi), \tag{31}$$

$$(\mathrm{id} \otimes \phi)\sigma = (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\phi \otimes \mathrm{id}). \tag{32}$$

Proof: Using (4) and (17) we obtain

$$\begin{aligned}(\epsilon \otimes \mathrm{id}^3)(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi) &= (\tau^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\\ = (\epsilon \otimes \mathrm{id} \otimes \sigma)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\phi \otimes \mathrm{id}) &= (\mathrm{id} \otimes \sigma)(\phi \otimes \mathrm{id})\sigma^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathrm{id}^3 \otimes \epsilon)(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\phi \otimes \mathrm{id}) &= (\mathrm{id} \otimes \tau^{-1})(\phi \otimes \mathrm{id})\\ = (\sigma \otimes \mathrm{id} \otimes \epsilon)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi) &= (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{-1}.\end{aligned}$$

Hence (29)–(30) hold. Starting from equalities (8) and (16) and applying the same computation we obtain (31)–(32). ∎

In the next proposition 'pentagonal' twisting relations including only τ are collected.

PROPOSITION 4: *We have*

$$(\phi \otimes \mathrm{id})\tau = (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi), \tag{33}$$

$$(\mathrm{id} \otimes \phi)\tau = (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\phi \otimes \mathrm{id}), \tag{34}$$

$$\tau(m \otimes \mathrm{id}) = (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes m), \tag{35}$$

$$\tau(\mathrm{id} \otimes m) = (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(m \otimes \mathrm{id}). \tag{36}$$

Proof: Direct transformations give

$$(\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi) = (\mathrm{id} \otimes \tau\sigma^{-1})(\phi \otimes \mathrm{id})\sigma = (\phi \otimes \mathrm{id})\tau.$$

Similarly,

$$(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\phi \otimes \mathrm{id}) = (\tau\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma = (\mathrm{id} \otimes \phi)\tau.$$

Applying (16), (31) and (1) we obtain

$$\begin{aligned}
(\mathrm{id}\otimes m)(\tau\otimes \mathrm{id})(\mathrm{id}\otimes\tau) &= (\mathrm{id}\otimes m\otimes\epsilon)(\tau\otimes\sigma^{-1})(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&= (\mathrm{id}\otimes m\otimes\epsilon)(\mathrm{id}^2\otimes\sigma^{-1})(\mathrm{id}\otimes\sigma^{-1}\otimes\mathrm{id})(\phi\otimes\mathrm{id}^2)\\
&\qquad (\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&= (\mathrm{id}^2\otimes\epsilon)(\mathrm{id}\otimes\sigma^{-1})(\phi\otimes m)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&= (\mathrm{id}^2\otimes\epsilon)(\mathrm{id}\otimes\sigma^{-1})(\phi\otimes\mathrm{id})\sigma(m\otimes\mathrm{id})\\
&= \tau(m\otimes\mathrm{id}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(m\otimes \mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id}) &= (\epsilon\otimes m\otimes\mathrm{id})(\sigma^{-1}\otimes\tau)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\sigma\otimes\mathrm{id})\\
&= (\epsilon\otimes m\otimes\mathrm{id})(\sigma^{-1}\otimes\mathrm{id}^2)(\mathrm{id}\otimes\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}^2\otimes\phi)\\
&\qquad (\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})\\
&= (\epsilon\otimes\mathrm{id}^2)(\sigma^{-1}\otimes\mathrm{id})(m\otimes\phi)(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})\\
&= \tau(\mathrm{id}\otimes m). \qquad \blacksquare
\end{aligned}$$

We pass to the study of algebraic relations including the antipode map. In the standard theory, the antipode is an anti(co)multiplicative map. The next proposition gives a braided counterpart of this property.

PROPOSITION 5: *We have*

$$\phi\kappa = \sigma(\kappa\otimes\kappa)\phi, \tag{37}$$

$$\kappa m = m(\kappa\otimes\kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau. \tag{38}$$

Proof: Let us start from the identity

$$\kappa(a^{(1)})a^{(2)}\otimes a^{(3)} = 1\otimes a.$$

Acting by $\phi\otimes\phi$ on both sides, and using (3) and (10), we obtain

$$\big(\phi\kappa(a^{(1)})\big)\big(a^{(2)}\otimes a^{(3)}\big)\otimes a^{(4)}\otimes a^{(5)} = 1\otimes 1\otimes a^{(1)}\otimes a^{(2)}.$$

After the action of $(\mathrm{id}\otimes m\otimes\mathrm{id})(\mathrm{id}^2\otimes\kappa\otimes\mathrm{id})$ on both sides the above equality becomes

$$\big(\phi\kappa(a^{(1)})\big)\big(a^{(2)}\otimes 1\big)\otimes a^{(3)} = 1\otimes\kappa(a^{(1)})\otimes a^{(2)}.$$

Hence

$$\big(\phi\kappa(a^{(1)})\big)\big(a^{(2)}\kappa(a^{(3)})\otimes 1\big)=\big(1\otimes\kappa(a^{(1)})\big)\big(\kappa(a^{(2)})\otimes 1\big).$$

Applying (5)–(6) we obtain

$$\phi\kappa(a)=\sigma\big(\kappa(a^{(1)})\otimes\kappa(a^{(2)})\big).$$

This proves (37). To prove (38), let us start from $m(\kappa\otimes m)(\phi\otimes \mathrm{id})=\epsilon\otimes \mathrm{id}$, act by it on $m\otimes m$, and apply (3) and (28). We find

$$m(\kappa\otimes m)(m\otimes m\otimes m)(\mathrm{id}\otimes\sigma\otimes \mathrm{id}^3)(\phi\otimes\phi\otimes \mathrm{id}^2)=(\epsilon\otimes\epsilon)\sigma^{-1}\tau\otimes m.$$

Acting by this equality on $(\mathrm{id}^2\otimes\kappa\otimes \mathrm{id})(\mathrm{id}\otimes\phi\otimes \mathrm{id})$ and simplifying the expression we find

$$m(\kappa m\otimes m)(\mathrm{id}\otimes\sigma\otimes \mathrm{id})(\phi\otimes \mathrm{id}^2)=(\epsilon\otimes m)(\mathrm{id}\otimes\kappa\otimes \mathrm{id})(\sigma^{-1}\tau\otimes \mathrm{id}).$$

Acting by this on $(\mathrm{id}^2\otimes\kappa)(\mathrm{id}\otimes\sigma)(\phi\otimes \mathrm{id})$ we obtain

$$\begin{aligned}&m(\kappa m\otimes m)(\mathrm{id}\otimes\sigma\otimes\kappa)(\mathrm{id}^2\otimes\sigma)(\mathrm{id}\otimes\phi\otimes \mathrm{id})(\phi\otimes \mathrm{id})\\&=(\epsilon\otimes m)(\mathrm{id}\otimes\kappa\otimes\kappa)(\sigma^{-1}\tau\otimes \mathrm{id})(\mathrm{id}\otimes\sigma)(\phi\otimes \mathrm{id}).\end{aligned}$$

After simple twisting transformations the left-hand side of the above equality becomes

$$\begin{aligned}&m(\kappa m\otimes m)(\mathrm{id}^3\otimes\kappa)(\mathrm{id}^2\otimes\phi)(\mathrm{id}\otimes\sigma\tau^{-1}\sigma)(\phi\otimes \mathrm{id})\\&=(\kappa m\otimes\epsilon)(\mathrm{id}\otimes\sigma\tau^{-1}\sigma)(\phi\otimes \mathrm{id})=\kappa m\tau^{-1}\sigma\tau^{-1}\sigma.\end{aligned}$$

The right-hand side of the mentioned equality reduces to

$$m(\epsilon\otimes\kappa\otimes\kappa)(\sigma^{-1}\otimes \mathrm{id})(\mathrm{id}\otimes\phi)\sigma=m(\kappa\otimes\kappa)\tau.$$

Consequently, (38) holds. ∎

Twisting properties of the antipode will be now analyzed. But first, a technical lemma.

LEMMA 6: *We have*

$$[\sigma(\kappa\otimes\mathrm{id})\tau^{-1}\sigma\tau^{-1}(a\otimes b^{(1)})]b^{(2)}=a\otimes 1\epsilon(b), \tag{39}$$

$$a^{(1)}[\sigma(\mathrm{id}\otimes\kappa)\tau^{-1}\sigma\tau^{-1}(a^{(2)}\otimes b)]=\epsilon(a)1\otimes b, \tag{40}$$

for each $a,b\in\mathcal{A}$.

Proof: We compute

$$\begin{aligned}(\mathrm{id}\otimes m)&(\sigma\otimes\mathrm{id})(\kappa\otimes\mathrm{id}^2)(\tau^{-1}\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\\ &=(\mathrm{id}\otimes m)(\sigma\otimes\mathrm{id})(\kappa\otimes\mathrm{id}^2)(\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\sigma\tau^{-1}\\ &=(\mathrm{id}\otimes m)(\sigma\otimes\mathrm{id})(\kappa\otimes\sigma)(\phi\otimes\mathrm{id})\tau^{-1}=\sigma(m\otimes\mathrm{id})(\kappa\otimes\mathrm{id}^2)(\phi\otimes\mathrm{id})\tau^{-1}\\ &=\sigma(1\epsilon\otimes\mathrm{id})\tau^{-1}\\ &=\mathrm{id}\otimes 1\epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}(m\otimes\mathrm{id})&(\mathrm{id}\otimes\sigma)(\mathrm{id}^2\otimes\kappa)(\mathrm{id}\otimes\tau^{-1}\sigma\tau^{-1})(\phi\otimes\mathrm{id})\\ &=(m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\mathrm{id}^2\otimes\kappa)(\mathrm{id}\otimes\tau^{-1})(\phi\otimes\mathrm{id})\sigma\tau^{-1}\\ &=\sigma(\mathrm{id}\otimes m)(\mathrm{id}\otimes\kappa)(\mathrm{id}\otimes\phi)\tau^{-1}=\sigma(\mathrm{id}\otimes 1\epsilon)\tau^{-1}=1\epsilon\otimes\mathrm{id}.\end{aligned}$$ ∎

PROPOSITION 7: *The following identities hold:*

$$\sigma(\kappa\otimes\mathrm{id})=(\mathrm{id}\otimes\kappa)\tau\sigma^{-1}\tau, \tag{41}$$

$$\tau(\mathrm{id}\otimes\kappa)=(\kappa\otimes\mathrm{id})\tau, \tag{42}$$

$$\tau(\kappa\otimes\mathrm{id})=(\mathrm{id}\otimes\kappa)\tau, \tag{43}$$

$$\sigma(\mathrm{id}\otimes\kappa)=(\kappa\otimes\mathrm{id})\tau\sigma^{-1}\tau. \tag{44}$$

Proof: Applying Lemma 6 and property (5) we obtain

$$\sigma(\kappa\otimes\mathrm{id})\tau^{-1}\sigma\tau^{-1}(a\otimes b)=[\sigma(\kappa\otimes\mathrm{id})\tau^{-1}\sigma\tau^{-1}(a\otimes b^{(1)})]b^{(2)}\kappa(b^{(3)})=a\otimes\kappa(b).$$

Similarly,

$$(\mathrm{id}\otimes\kappa)\tau^{-1}\sigma\tau^{-1}(a\otimes b)=\kappa(a^{(1)})a^{(2)}[\sigma(\kappa\otimes\mathrm{id})\tau^{-1}\sigma\tau^{-1}(a^{(3)}\otimes b)]=\kappa(a)\otimes b.$$

This shows (41) and (44). Using properties (16), (41), (44), (22)–(23) and (33)–(34) we obtain

$$\begin{aligned}\tau(\mathrm{id}\otimes\kappa)&=(\epsilon\otimes\mathrm{id}^2)(\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\sigma(\mathrm{id}\otimes\kappa)\\ &=(\epsilon\otimes\kappa\otimes\mathrm{id})(\tau^{-1}\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\tau\sigma^{-1}\tau\\ &=(\epsilon\otimes\kappa\otimes\mathrm{id})(\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\tau=(\kappa\otimes\mathrm{id})\tau.\end{aligned}$$

Similarly,

$$\begin{aligned}\tau(\kappa\otimes\mathrm{id}) &= (\mathrm{id}^2\otimes\epsilon)(\mathrm{id}\otimes\sigma^{-1})(\phi\otimes\mathrm{id})\sigma(\kappa\otimes\mathrm{id})\\ &= (\mathrm{id}\otimes\kappa\otimes\epsilon)(\mathrm{id}\otimes\tau^{-1}\sigma\tau^{-1})(\phi\otimes\mathrm{id})\tau\sigma^{-1}\tau\\ &= (\mathrm{id}\otimes\kappa\otimes\epsilon)(\mathrm{id}\otimes\tau^{-1})(\phi\otimes\mathrm{id})\tau = (\mathrm{id}\otimes\kappa)\tau. \quad\blacksquare\end{aligned}$$

As a direct consequence of the previous proposition we find

$$(\kappa\otimes\kappa)\tau = \tau(\kappa\otimes\kappa), \tag{45}$$

$$(\kappa\otimes\kappa)\sigma = \sigma(\kappa\otimes\kappa). \tag{46}$$

To end this section, we shall prove that σ and τ satisfy a system of braid equations.

PROPOSITION 8: *The following identities hold:*

$$(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id}) = (\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma), \tag{47}$$

$$(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id}) = (\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau), \tag{48}$$

$$(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id}) = (\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma), \tag{49}$$

$$(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id}) = (\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma), \tag{50}$$

$$(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id}) = (\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau), \tag{51}$$

$$(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id}) = (\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau), \tag{52}$$

$$(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id}) = (\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma), \tag{53}$$

$$(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id}) = (\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau). \tag{54}$$

Proof: We shall first prove (48)–(51) and (53), secondly (54), thirdly (52) and finally (47). A direct computation gives

$$\begin{aligned}&(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})\\ &= (\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(m\otimes m\otimes\mathrm{id})(\kappa\otimes\phi m\otimes\kappa\otimes\mathrm{id})(\phi\otimes\phi\otimes\mathrm{id})\\ &= A(\tau\otimes\mathrm{id}^3)(\mathrm{id}\otimes\tau\otimes\mathrm{id}^2)(\mathrm{id}^2\otimes\sigma\otimes\mathrm{id})(\mathrm{id}^3\otimes\sigma)(\kappa\otimes\phi m\otimes\kappa\otimes\mathrm{id})(\phi\otimes\phi\otimes\mathrm{id})\\ &= A(\mathrm{id}\otimes\kappa\otimes\phi\otimes\kappa)(\tau\otimes\mathrm{id}^2)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\mathrm{id}\otimes m\otimes\tau\sigma^{-1}\tau)(\phi\otimes\phi\otimes\mathrm{id})\\ &= (\mathrm{id}\otimes m\otimes m)(\mathrm{id}\otimes\kappa\otimes\phi m\otimes\kappa)(\tau\otimes\mathrm{id}^3)(\mathrm{id}\otimes\sigma\otimes\mathrm{id}^2)(\phi\otimes\mathrm{id}\otimes\phi)(\mathrm{id}\otimes\tau)\\ &= (\mathrm{id}\otimes m\otimes m)(\mathrm{id}\otimes\kappa\otimes\phi m\otimes\kappa)(\mathrm{id}\otimes\phi\otimes\phi)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\ &= (\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau), \qquad \text{where } A = \mathrm{id}\otimes m\otimes m.\end{aligned}$$

Similarly,

$$
\begin{aligned}
&(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes m\otimes m)(\mathrm{id}\otimes\kappa\otimes\phi m\otimes\kappa)(\mathrm{id}\otimes\phi\otimes\phi)\\
&=B(\mathrm{id}^3\otimes\tau)(\mathrm{id}^2\otimes\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma\otimes\mathrm{id}^2)(\sigma\otimes\mathrm{id}^3)(\mathrm{id}\otimes\kappa\otimes\phi m\otimes\kappa)(\mathrm{id}\otimes\phi\otimes\phi)\\
&=B(\kappa\otimes\phi\otimes\kappa\otimes\mathrm{id})(\mathrm{id}^2\otimes\tau)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\tau\sigma^{-1}\tau\otimes m\otimes\mathrm{id})(\mathrm{id}\otimes\phi\otimes\phi)\\
&=(m\otimes m\otimes\mathrm{id})(\kappa\otimes\phi m\otimes\kappa\otimes\mathrm{id})(\mathrm{id}^3\otimes\tau)(\mathrm{id}^2\otimes\sigma\otimes\mathrm{id})(\phi\otimes\mathrm{id}\otimes\phi)(\tau\otimes\mathrm{id})\\
&=(m\otimes m\otimes\mathrm{id})(\kappa\otimes\phi m\otimes\kappa\otimes\mathrm{id})(\phi\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id})\\
&=(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id}),\qquad\text{where } B=m\otimes m\otimes\mathrm{id}.
\end{aligned}
$$

Essentially the same transformations lead to identities (49), (51) and (53). Let us prove (54). We have

$$
\begin{aligned}
(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)&=(\mathrm{id}\otimes\tau\otimes\epsilon)(\tau\otimes\sigma^{-1})(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\mathrm{id}^2\otimes\epsilon\otimes\mathrm{id})(\mathrm{id}^2\otimes\tau)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\tau\otimes\sigma^{-1})\\
&\qquad(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\mathrm{id}^2\otimes\epsilon\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}^2\otimes\tau)(\mathrm{id}\otimes\tau\otimes\mathrm{id})\\
&\qquad(\tau\otimes\mathrm{id}^2)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\mathrm{id}^2\otimes\epsilon\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1}\otimes\mathrm{id})(\phi\otimes\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\tau\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&=(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id}).
\end{aligned}
$$

Identities (25), (48), (51) and (54) imply

$$
\begin{aligned}
&(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\
&\quad=(\mathrm{id}\otimes\tau)(\sigma\tau^{-1}\otimes\mathrm{id})(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\
&\quad=(\mathrm{id}\otimes\sigma)(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1}\tau)(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\
&\quad=(\mathrm{id}\otimes\sigma)(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^{-1})(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\tau\otimes\mathrm{id})\\
&\quad=(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\sigma^{-1}\tau\otimes\mathrm{id})\\
&\quad=(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id}).
\end{aligned}
$$

Finally (24), (48), (50) and (52) imply

$$\begin{aligned}
&(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&\quad=(\mathrm{id}\otimes\sigma\tau^{-1})(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\tau\otimes\mathrm{id})\\
&\quad=(\mathrm{id}\otimes\sigma\tau^{-1})(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\tau)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\
&\quad=(\sigma\tau^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\tau)\\
&\quad=(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id}). \qquad \blacksquare
\end{aligned}$$

Appendix A. Braid systems

The presence of two different braid operators σ and τ in the twisting properties of ϕ and κ implies that, in contrast to the standard formalism [6], the theory is not includable in the conceptual framework of braided categories. In this appendix we shall prove that σ and τ can be included in a generally infinite system of braid operators indexed by integers, expressing all twisting properties in a concise and elegant way. Finally, we give a characterization of the standard theory, in terms of the multiplicativity of the counit map.

Let us consider a complex associative algebra $\mathcal{A}$ with the unit element $1\in\mathcal{A}$ and the product $m\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$.

Definition 2: A **braid system** over $\mathcal{A}$ is a collection $\mathcal{F}$ of bijective linear maps acting in $\mathcal{A}\otimes\mathcal{A}$ and satisfying

$$(\alpha\otimes\mathrm{id})(\mathrm{id}\otimes\beta)(\gamma\otimes\mathrm{id})=(\mathrm{id}\otimes\gamma)(\beta\otimes\mathrm{id})(\mathrm{id}\otimes\alpha), \tag{55}$$

$$\alpha(\mathrm{id}\otimes m)=(m\otimes\mathrm{id})(\mathrm{id}\otimes\alpha)(\alpha\otimes\mathrm{id}), \tag{56}$$

$$\alpha(m\otimes\mathrm{id})=(\mathrm{id}\otimes m)(\alpha\otimes\mathrm{id})(\mathrm{id}\otimes\alpha), \tag{57}$$

for each $\alpha,\beta,\gamma\in\mathcal{F}$.

Definition 3: A braid system $\mathcal{F}$ is called **complete** iff it is closed under the operation $(\alpha,\beta,\gamma)\mapsto\alpha\beta^{-1}\gamma$.

Let $\mathcal{F}$ be a braid system over $\mathcal{A}$. Then

$$\alpha\bigl(1\otimes(\,)\bigr)=(\,)\otimes1,\qquad \alpha\bigl((\,)\otimes1\bigr)=1\otimes(\,)$$

for each $\alpha\in\mathcal{F}$, as follows from (56)–(57). Further, every $\alpha\in\mathcal{F}$ naturally determines an associative algebra structure on $\mathcal{A}\otimes\mathcal{A}$, with the unit element $1\otimes1$. The corresponding product is given by $(m\otimes m)(\mathrm{id}\otimes\alpha\otimes\mathrm{id})$.

We are going to prove that there exists the *minimal* complete braid system $\mathcal{F}^*$ which extends $\mathcal{F}$. Starting from the system $\mathcal{F}$ we can inductively construct an increasing chain of braid systems $\mathcal{F}_n$, where $n \geq 0$ and $\mathcal{F}_0 = \mathcal{F}$, while $\mathcal{F}_{n+1}$ consists of maps of the form $\delta = \alpha\beta^{-1}\gamma$, where $\alpha, \beta, \gamma \in \mathcal{F}_n$. The fact that all $\mathcal{F}_n$ are braid systems easily follows by induction, applying the definition of braid systems and the identity

$$(\alpha\beta^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \gamma\delta^{-1}) = (\mathrm{id} \otimes \gamma\delta^{-1})(\alpha\beta^{-1} \otimes \mathrm{id}) \tag{58}$$

(which holds in an arbitrary braid system).

Let $\mathcal{F}^*$ be the union of systems $\mathcal{F}_n$. By construction, $\mathcal{F}^*$ is a complete braid system. Moreover, $\mathcal{F}^*$ is the minimal braid system containing $\mathcal{F}$.

Let $G = (\mathcal{A}, \{\phi, \epsilon, \kappa, \sigma\})$ be a braided quantum group. According to (1)–(2), (35)–(36) and Proposition 8 operators $\{\sigma, \tau\}$ form a braid system over the algebra $\mathcal{A}$. The corresponding completion $\mathcal{F} = \{\sigma, \tau\}^*$ consists of maps $\sigma_n\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of the form

$$\sigma_n = (\sigma\tau^{-1})^{n-1}\sigma = \sigma(\tau^{-1}\sigma)^{n-1} \tag{59}$$

where $n \in \mathbb{Z}$.

PROPOSITION 9: *The following identities hold:*

$$(\phi \otimes \mathrm{id})\sigma_{n+k} = (\mathrm{id} \otimes \sigma_k)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \phi), \tag{60}$$

$$\sigma_n(\mathrm{id} \otimes \kappa) = (\kappa \otimes \mathrm{id})\sigma_{-n}, \tag{61}$$

$$\sigma_n(\kappa \otimes \mathrm{id}) = (\mathrm{id} \otimes \kappa)\sigma_{-n}, \tag{62}$$

$$(\mathrm{id} \otimes \phi)\sigma_{n+k} = (\sigma_k \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\phi \otimes \mathrm{id}). \tag{63}$$

Proof: Applying Proposition 7 and (59) we obtain

$$\begin{aligned}\sigma_n(\mathrm{id} \otimes \kappa) &= (\sigma\tau^{-1})^{n-1}\sigma(\mathrm{id} \otimes \kappa) = (\kappa \otimes \mathrm{id})(\tau\sigma^{-1})^{n-1}\tau\sigma^{-1}\tau\\ &= (\kappa \otimes \mathrm{id})(\sigma\tau^{-1})^{-n-1}\sigma = (\kappa \otimes \mathrm{id})\sigma_{-n}.\end{aligned}$$

Similarly,

$$\sigma_n(\kappa \otimes \mathrm{id}) = (\mathrm{id} \otimes \kappa)(\tau\sigma^{-1})^{n-1}\tau\sigma^{-1}\tau = (\mathrm{id} \otimes \kappa)\sigma_{-n}.$$

Equalities (60) and (63) follow directly from (20)–(23) and (29)–(30). Indeed,

$$\begin{aligned}(\sigma_k \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\phi \otimes \mathrm{id}) &= \big((\sigma\tau^{-1})^k\tau \otimes \mathrm{id}\big)\big(\mathrm{id} \otimes \sigma(\tau^{-1}\sigma)^{n-1}\big)(\phi \otimes \mathrm{id})\\ &= \big((\sigma\tau^{-1})^k\tau \otimes \mathrm{id}\big)(\mathrm{id} \otimes \sigma)(\phi \otimes \mathrm{id})(\tau^{-1}\sigma)^{n-1}\\ &= \big((\sigma\tau^{-1})^k \otimes \mathrm{id}\big)(\mathrm{id} \otimes \phi)\sigma(\tau^{-1}\sigma)^{n-1}\\ &= (\mathrm{id} \otimes \phi)(\sigma\tau^{-1})^{n+k-1}\sigma = (\mathrm{id} \otimes \phi)\sigma_{n+k}.\end{aligned}$$

Similarly,

$$(\mathrm{id}\otimes\sigma_n)(\sigma_k\otimes\mathrm{id})(\mathrm{id}\otimes\phi)=\big(\mathrm{id}\otimes(\sigma\tau^{-1})^n\tau\big)\big(\sigma(\tau^{-1}\sigma)^{k-1}\otimes\mathrm{id}\big)(\mathrm{id}\otimes\phi)$$
$$=(\phi\otimes\mathrm{id})(\sigma\tau^{-1})^{n+k-1}\sigma=(\phi\otimes\mathrm{id})\sigma_{n+k}. \qquad \blacksquare$$

As we shall now see, an arbitrary $\sigma_n\in\mathcal{F}$ is interpretable as the flip-over operator corresponding to a modified braided quantum group structure.

For each $n\in\mathbb{Z}$, let $m_n\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ and $\kappa_n\colon\mathcal{A}\to\mathcal{A}$ be the maps given by

$$m_n=m\sigma_n^{-1}\sigma, \tag{64}$$

$$\kappa_n=(\epsilon\otimes\kappa)\sigma_n^{-1}\sigma\phi=(\kappa\otimes\epsilon)\sigma_n^{-1}\sigma\phi \tag{65}$$

(the second equality in (65) will be justified in the proof of the proposition below). It is easy to see that each m_n, interpreted as a product, determines a structure of an associative algebra on the space $\mathcal{A}$. Indeed,

$$\begin{aligned}
m_n(m_n\otimes\mathrm{id})&=m\sigma_n^{-1}\sigma(m\sigma_n^{-1}\sigma\otimes\mathrm{id})\\
&=m\sigma_n^{-1}(\mathrm{id}\otimes m)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma_n^{-1}\sigma\otimes\mathrm{id})\\
&=m(m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1})(\sigma_n^{-1}\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma_n^{-1}\sigma\otimes\mathrm{id})\\
&=m(m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1})(\sigma_n^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1})(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\\
&\qquad(\sigma\otimes\mathrm{id})\\
&=m(\mathrm{id}\otimes m)(\sigma_n^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1})(\sigma_n^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})\\
&\qquad(\mathrm{id}\otimes\sigma)\\
&=m(\mathrm{id}\otimes m)(\sigma_n^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1}\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1}\sigma)\\
&=m\sigma_n^{-1}(m\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_n^{-1}\sigma)\\
&=m\sigma_n^{-1}\sigma(\mathrm{id}\otimes m\sigma_n^{-1}\sigma)\\
&=m_n(\mathrm{id}\otimes m_n).
\end{aligned}$$

For each $n\in\mathbb{Z}$, let us denote by $\mathcal{A}_n$ the vector space $\mathcal{A}$ endowed with the product m_n. Evidently, $1\in\mathcal{A}_n$ is the unit in this algebra, too.

PROPOSITION 10: *The pair $G_n=\big(\mathcal{A}_n,\{\phi,\epsilon,\kappa_n,\sigma_n\}\big)$ is a braided quantum group.*

Proof: We have to check the last three axioms in Definition 1. The compatibility condition between ϕ and σ_n follows easily from (60) and (63). Further, a direct computation gives

$$\begin{aligned}\phi m_n &= (m\otimes m)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\phi\otimes\phi)\sigma_n^{-1}\sigma\\ &= (m\otimes m)(\mathrm{id}\otimes\sigma\sigma_n^{-1}\sigma\otimes\mathrm{id})(\phi\otimes\phi)\\ &= (m\otimes m)(\mathrm{id}\otimes\sigma_{2-n}\otimes\mathrm{id})(\phi\otimes\phi)\\ &= (m\sigma_n^{-1}\otimes m)(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\sigma_2\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\\ &= (m\sigma_n^{-1}\sigma\otimes m)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\phi\otimes\phi)\\ &= (m\sigma_n^{-1}\sigma\otimes m\sigma_n^{-1})(\mathrm{id}\otimes\phi\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{n+1})(\phi\otimes\mathrm{id})\\ &= (m_n\otimes m_n)(\mathrm{id}\otimes\sigma_n\otimes\mathrm{id})(\phi\otimes\phi).\end{aligned}$$

Finally, we have to check that k_n satisfies the antipode axiom. Let us consider maps $k_n^{\pm}\colon\mathcal{A}\to\mathcal{A}$ given by

$$k_n^- = (\kappa\otimes\epsilon)\sigma_n^{-1}\sigma\phi,\qquad k_n^+ = (\epsilon\otimes\kappa)\sigma_n^{-1}\sigma\phi.$$

We have

$$\begin{aligned}m_n(k_n^-\otimes\mathrm{id})\phi &= m\sigma_n^{-1}\sigma(\epsilon\otimes\kappa\otimes\mathrm{id})(\tau\sigma_n^{-1}\sigma\otimes\mathrm{id})(\phi\otimes\mathrm{id})\phi\\ &= m(\epsilon\otimes\kappa\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{-n}^{-1}\sigma_{-1})(\sigma_{1-n}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\phi\\ &= m(\epsilon\otimes\kappa\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{-n}^{-1})(\phi\otimes\mathrm{id})\sigma_{-n}\phi\\ &= m(\epsilon\otimes\kappa\otimes\mathrm{id})(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\phi\\ &= m(\kappa\otimes\mathrm{id})\phi = 1\epsilon.\end{aligned}$$

Similarly, it follows that $m_n(\mathrm{id}\otimes\kappa_n^+)\phi = 1\epsilon$. To complete the proof, let us observe that

$$\begin{aligned}\kappa_n^+ = (\epsilon\otimes\kappa_n^+)\phi &= m_n(m_n\otimes\mathrm{id})(\kappa_n^-\otimes\mathrm{id}\otimes\kappa_n^+)(\phi\otimes\mathrm{id})\phi\\ &= m_n(\mathrm{id}\otimes m_n)(\kappa_n^-\otimes\mathrm{id}\otimes\kappa_n^+)(\mathrm{id}\otimes\phi)\phi = m_n(\kappa_n^-\otimes 1\epsilon)\phi = k_n^-.\end{aligned}$$

The map $k_n = k_n^{\pm}$ is bijective. Its inverse is given by

$$\kappa_n^{-1}\kappa = (\epsilon\otimes\mathrm{id})\sigma^{-1}\sigma_n\phi = (\mathrm{id}\otimes\epsilon)\sigma^{-1}\sigma_n\phi. \qquad\blacksquare$$

From the point of view of this analysis, the group G_0 is particularly interesting. For example, left-covariant first-order differential structures over G (braided counterparts of structures considered in [8]) are in a natural bijection with certain right $\mathcal{A}_0$-ideals $\mathcal{R}\subseteq\ker(\epsilon)$. Informally speaking, G_0 is interpretable as a 'maximal braided simplification' of G, with the same coalgebra structure. It is a standard braided-Hopf algebra.

If G is a standard braided-Hopf algebra then the counit is multiplicative. Interestingly, the converse is also true.

LEMMA 11: *The following properties are equivalent:*

$$\epsilon m = \epsilon \otimes \epsilon, \tag{66}$$

$$(\epsilon \otimes \mathrm{id})\sigma = \mathrm{id} \otimes \epsilon, \tag{67}$$

$$(\mathrm{id} \otimes \epsilon)\sigma = \epsilon \otimes \mathrm{id}, \tag{68}$$

$$\sigma = \tau. \tag{69}$$

Proof: Equality (69) implies (66), according to (28). If (66) holds then (13)–(14) imply (67)–(68). Finally, if (67) (or (68)) holds, (16) implies that two flip-over operators coincide. ∎

In other words, the above-listed conditions characterize the theory of [6]. Indeed, $\sigma = \tau$ implies that the whole system $\mathcal{F}$ reduces to a single braiding σ, and all maps appearing in the game are understandable as morphisms in a braided category generated by $\mathcal{A}$ and σ.

In the standard theory [6] all computations can be performed diagramatically, drawing braid and tangle diagrams. A similar situation holds here, in the general multi-braided framework. The only difference is that diagrams should be appropriately refined, by identifying each separate braiding (labeling them by integers) and properly expressing all the derived algebraic properties.

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